Classical Lie Groups, Quantum Groups, and Special Functions

dedicated to Prof. T.H. Koornwinder

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A brief overview of the significance of representations of classical Lie groups and of quantum groups in the theory of special functions is given. There are many directions of interrelations between group representations and special functions. We are mainly concerned with those of them which are connected with results obtained by Prof. T.H. Koornwinder, CWI and University of Amsterdam. Namely, we consider Koornwinder's addition and product formulas for Jacobi and Laguerre polynomials, the group theoretical approach to polynomials of a discrete variable, generalizations of the hypergeometric function related to root systems, special functions of a matrix variable, and the approach to q-special functions by means of representations of quantum groups.

1. Introduction

The contemporary theory of special functions became a wide part of mathematics and it is of great importance for different branches of science and technology. Books containing hundreds of pages were devoted to studying separate classes of special functions. The fullest account of the results on special functions obtained by the middle of the 20-th century is given by the five-volume collection published by the "Bateman Project" (three volumes of "Higher Transcendental Functions" and two volumes of "Integral Transforms").

Different mathematical methods (the theory of differential equations, analytical methods, integral transforms and so on) are used in the theory of special functions. However, a really unified view on the theory of main classes of special functions was established by using the theory of group representations. Representations of Lie groups are of utmost importance for special functions. Lie groups can be considered as a bridge between special functions and geometry. Really, Lie groups are related to symmetries and homogeneity of some objects in multidimensional geometry. The objects include, in particular, spheres, hyperboloids, and paraboloids in multidimensional spaces (real, complex, quaternion, and octave ones) as well as their generalizations such as Stiefel manifolds, homogeneous cones, homogeneous complex regions, and so on. For these manifolds there are transformation Lie groups G which act transitively upon them and for any pair of points they allow a symmetry (permuting these points) with respect to a G-invariant metric. The spaces characterized by these properties are called symmetric. Riemannian symmetric spaces were introduced and studied by E. Cartan.

The relationship between special functions and the geometry of homogeneous spaces is based on the following facts. The special functions most often arise when the equations of mathematical physics are solved by the method of separation of variables in a certain coordinate system. The most important equations are invariant under some transformation groups (for example, the Laplace equation is invariant under the group of motions of Euclidean space $\mathbb{R}^n$, the wave equation under the group of linear transformations that preserve a quadratic form $x_0^2 - x_1^2 - \ldots - x_n^2$, the Maxwell equation under the Poincaré group, etc.). But the Laplace operator coincides, up to a constant factor, with the operator $\lim_{r\to 0}[S(x,r,f) - f(x)]/r^2$, where $S(x,r,f)$ is the mean value of the function f on the sphere with center x and radius r. It can be therefore defined in a natural way on symmetric spaces, giving on them G-invariant differential operators. This allows us to construct on such spaces the analogues of the classical differential equations of mathematical physics. When the variables are separated the spaces are fibered into coordinate surfaces which, in turn, are symmetric spaces. The special functions arise when eigenfunctions of invariant differential operators (in particular, of the Laplace operator and its generalizations) are sought, and it is therefore clear that their properties should involve the invariance of the operators under transformations of the group G. Any eigenfunction of an invariant operator transforms under the action of $g \in G$ into an eigenfunction that corresponds to the same eigenvalue. The linear transformation $T(g)$ is thus defined in the space of such eigenfunctions, and here the equality $T(g_1)T(g_2) = T(g_1g_2)$ is valid. The correspondence $g \to T(g)$ is a representation of the group G. So, we throw a bridge between the differential operators invariant under the action of some group G and representations of this group, which is also connected with special functions.

Hence, the group theoretical approach to special functions involves different branches of mathematics: differential operators, differential equations, geometry, and so on. A recent development of the subject shows that quantum groups (new mathematical objects which appeared in the theory of quantum inverse scattering) are very useful for studying q-orthogonal polynomials and basic hypergeometric functions.

In this paper we are concerned with some of the main directions of interrelations between group representations and special functions. These directions are related to contribution of Tom Koornwinder to the subject. Because of limited size of this article, we do not consider many interesting group theoretical methods in studying special functions. In particular, we are not dealing with the role of representations of symmetric groups in the theory of polynomials of discrete variables [5, 6] and of representations of Chevalley groups in the theory of q-orthogonal polynomials [38-40], we do not show applications of infinite parameter Lie groups in the theory of special functions [16, 41]. Let us note that these and all other main directions of applications of group representations in the theory of special functions are presented in the book [45].

2. MATRIX ELEMENTS OF GROUP REPRESENTATIONS AND SPECIAL FUNCTIONS

In fact, applications of group theoretical methods to special functions go back to E. Cartan. He developed a general theory of zonal spherical functions on compact symmetric spaces $X = G/H$, i.e. the functions that are constant under the action of a stationary subgroup H of some point and such that their shifts generate a subspace in which an irreducible representation of this group G is realized. If $G = SO(3)$, $H = SO(2)$, then X is a sphere, and zonal spherical functions coincide with the classical polynomials introduced by Legendre and Laplace. A system of zonal spherical functions is orthogonal with respect to an invariant measure on X. A similar theory is constructed on locally compact symmetric Riemannian spaces, but then the set of zonal spherical functions has the cardinality of the continuum and their orthogonality is interpreted in the same sense as in the theory of Fourier integrals. For example, in the case when X is a two-sheeted hyperboloid we obtain a set of Legendre functions. The methods used by Cartan were based on the ideas employed by H. Weyl and F. Peter to prove the general theorem that matrix elements of irreducible unitary representations of a compact group G form a complete orthogonal set of functions on G. This theorem explains the orthogonality of many systems of special functions.

The strong stimulus for stydying the relationship between special functions and the theory of group representations was given by the development of physics. To solve differential equations available in quantum mechanics it was necessary to use the symmetry of the physical systems under study, i.e. transformation groups that leave invariant some important characteristics of these systems (for example, potential in the Schrödinger equation). Since for some particular cases (for example, for the harmonic oscillator) solutions of these equations could be expressed in terms of the special functions, it was necessary to establish a relationship between the theory of these functions and the transformation group that leaves invariant the physical systems studied. Here we must mention Wigner's contribution. In those years group theory was not known to physicists of classical school (astronomer J. Jince even said that the physicists would never need it), and this period in the development of theoretical physics was named "the Gruppenpest". Spectroscopy studies began to make increasingly wider use of such concepts as Clebsch-Gordan coefficients, Racah coefficients and more general symbols related to a decomposition into irreducible representations of the tensor product of representations of groups. The requirements of relativistic physics advanced the task of studying the representations of noncompact Lie groups, in particular, of the Lorentz group $SO_o(3,1)$ and of its three-dimensional analogue $SO_o(2,1)$. These investigations resulted in the theory of infinite dimesional representations of semisimple Lie groups (V. Bargmann, I. Gel'fand and M. Naimark, Harish-Chandra) and then of nilpotent (A. Kirilov) and solvable (Auslander) Lie groups.

While studying the matrix elements of irreducible unitary representations of the group $SO_o(2,1)$, V. Bargmann found that they are expressed in terms of hypergeometric functions. Moreover, the matrix elements of representations of the discrete series are expressed through a particular case of this function (Jacobi polynomials). The same polynomials are used to express the matrix elements of irreducible unitary representations of the group $SO(3)$ which is a compact real form of the group $SO(3,\mathbb{C})$, a complexification of the group $SO_o(2,1)$. "Straightening" the groups $SO_o(2,1)$ and $SO(3)$, we obtain the group $ISO(2)$ of motions of the Euclidean plane. The matrix elements of irreducible unitary representations of this group are expressed in terms of the Bessel function.

So, the theory of the classes of special functions most important for applications - the hypergeometric function and the Bessel function - turned out to be associated with the representations of the simplest noncompact Lie groups.

Cartan's theory of zonal spherical functions constructed earlier was associated with the matrix elements of representations of class 1, i.e. such that their space has a single vector $\mathbf{v}_0$ invariant under the operators $T(h)$, $h \in H$. If we take $\mathbf{v}_0$ to be one of the basis vectors, the corresponding matrix element $\langle T(g)\mathbf{v}_0, \mathbf{v}_0\rangle$ will be constant on two-sided coset spaces with respect to the subgroup H, and so it gives a zonal spherical function on $X = G/H$. Similarly, matrix elements such as $\langle T(g)\mathbf{v}_0, \mathbf{v}\rangle$ and $\langle T(g)\mathbf{v}, \mathbf{v}_0\rangle$ are expressed through associated spherical functions.

The fact, that matrix elements of group representations are expressed in terms of special functions, allows us to study properties of these functions. For example, representation of a matrix element in the form $t_{mn}(g) = \langle T(g)\mathbf{v}_n, \mathbf{v}_m\rangle$ gives integral formulas for special functions if carrier spaces of representations are spaces of functions. Moreover, equalities of the type $T(g_1)T(g_2) = T(g_1g_2)$, where T is a representation and $g_1, g_2 \in G$, lead to addition theorems for special functions. In a similar way we obtain recurrence relations, generating functions for special functions, and so on.

Considering matrix elements of group representations we obtain special functions with discrete values of parameters. In order to obtain the properties of arbitrary special functions it is necessary to choose continual bases composed of generalized functions of representation spaces (similar to the basis $\{e^{i\lambda x}\}$ in the space $L^2(\mathbb{R})$), for example, the bases that diagonalize a noncompact one-parameter subgroup. Operators of representations of the group $SO_o(2,1)$ in these bases are given by integral operators whose kernels are expressed in terms of the hypergeometric function or cylindrical Hankel functions. Applying for these kernels the methods used for usual matrix elements we obtain relations for these functions including "continual addition theorems" in which the integration is over the parameters of the functions, but not over arguments. Ordinary addition theorems are derived from them using residue theorems. It is of interest to consider "mixed bases", i.e. a decomposition of the result of action of representation operators upon elements of one basis into elements of the other one. As a result, in the case of the group $SO_o(2,1)$ we get the Whittaker functions, the Laguerre polynomials, the Pollaczek polynomials and different relations connecting these functions with the hypergeometric function. We note that the Whittaker functions and the Laguerre polynomials also appear when we study the matrix elements of irreducible representations of the group S of triangular third-order matrices which is an extension of the Heisenberg group, the simplest in the class of nilpotent groups [44]. A series of new relations for special functions arises when we realize the representations using boson creation and annihilation operators.

3. Koornwinder's addition theorems for Jacobi and Laguerre polynomials

Let $g_{i-1}(\theta)$ be the matrix of the group $SO(n)$ describing rotations in the plane $(i-1,i)$ by the angle θ. The relation

$$g_{n-1}(\theta)g_{n-2}(\psi)g_{n-1}(\varphi) = g_{n-2}(\alpha)g_{n-1}(\gamma)g_{n-2}(\beta) \tag{1}$$

is fulfilled in the group $SO(n)$, where the angles α, γ, β are expressed in terms of the angles θ, ψ, φ, in particular

$$\cos\gamma = \cos\theta\cos\varphi + \sin\theta\sin\varphi\cos\psi.$$

Irreducible unitary representations of $SO(n)$ which are of class 1 with respect to the subgroup $SO(n-1)$ are given by a nonnegative integer m. We denote these representations by T_m. For the representation T_m relation (1) can be written as

$$\begin{aligned} &T_m(g_{n-1}(\theta))T_m(g_{n-2}(\psi))T_m(g_{n-1}(\varphi)) \\ &\qquad = T_m(g_{n-2}(\alpha))T_m(g_{n-1}(\gamma))T_m(g_{n-2}(\beta)). \end{aligned} \tag{1'}$$

We take the basis of the carrier space of the representation T_m which corresponds to the successive restriction of T_m onto the subgroups $SO(n-1) \supset SO(n-2) \supset SO(n-3) \supset \dots \supset SO(2)$. The basis elements are given by integers $M \equiv (m, m_1, m_2, \dots m_{n-2})$ such that $m \geq m_1 \geq m_2 \geq \dots m_{n-2} \geq 0$. The matrix element

$$t^m_{MN}(g_{n-1}(\theta)) = \langle T_m(g_{n-1}(\theta))\mathbf{v}_N, \mathbf{v}_M\rangle$$

vanishes unless $m_i = n_i$, $i = 2,3,\dots,n-2$. If these equalities hold, then the matrix element does not depend on $m_3, m_4, \dots, m_{n-1}$. In particular, the zonal spherical function $t^m_{OO}(g_{n-1}(\theta))$ (where $M = N = O = (0,\dots,0)$) coincides with

$$t^m_{OO}(g_{n-1}(\theta)) = \frac{m!(n-3)!}{(n+m-3)!}C_m^{(n-2)/2}(\cos\theta),$$

where $C_n^\alpha(x)$ is the Gegenbauer polynomial. The associated spherical function $t^m_{m_1 0}(g_{n-1}(\theta)) \equiv t^m_{MO}(g_{n-1}(\theta))$ is equal to

$$t^m_{m_1 0}(g_{n-1}(\theta)) = c \sin^{m_1}\theta C^{m_1+(n-2)/2}_{m-m_1}(\cos\theta),$$

where c is a constant [43].

We obtain from formula (1′) the relation

$$\sum_{k=0}^{m} t^m_{0k}(g_{n-1}(\theta)) d^k_{00}(g_{n-2}(\psi)) t^m_{k0}(g_{n-1}(\varphi)) = t^m_{00}(g_{n-1}(\gamma)),$$

where $d^k_{00}(g_{n-2}(\psi))$ is the zonal spherical function of the representation T_k of the subgroup $SO(n-1)$. Substituting here expressions for matrix elements we directly obtain the well-known addition theorem for Gegenbauer polynomials [43].

If we use the class 1 irreducible representations of the group $U(n)$ instead of those for $SO(n)$ and the relation

$$g_{n-1}(\theta_1) g_{n-2}(\varphi) d_n(\psi) g_{n-1}(\theta_2) = k g_{n-1}(\theta) d_n(\psi_1) k', \tag{2}$$

where $d_n(\psi)$ is the diagonal matrix diag $(1, \dots 1, e^{i\psi})$ and k, k' are elements of the subgroup $U(n-1)$, then in the same way we obtain the addition theorem for Jacobi polynomials:

$$\begin{aligned} P^{(p,0)}_m(\cos 2\theta) &= \sum_{k=0}^{m} \sum_{l=0}^{k} a_{mkl} (\sin\theta_1 \sin\theta_2)^{k+l} (\cos\theta_1 \cos\theta_2)^{k-l} \\ &\times P^{(p+k+l,k-l)}_{m-k}(\cos 2\theta_1) P^{(p+k+l,k-l)}_{m-k}(\cos 2\theta_2) \\ &\times P^{(p-1,k-l)}_l(\cos 2\varphi) \cos^{k-l}\varphi \cos(k-l)\psi, \end{aligned} \tag{3}$$

where

$$\cos 2\theta = 2|\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\varphi\, e^{i\psi}|^2 - 1 \tag{3'}$$

and a_{mkl} are constants. Differentiating both sides with respect to $\cos\psi$ and taking into account the formula

$$\frac{d}{dx} P^{(\alpha,\beta)}_n(x) = \frac{1}{2}(\alpha+\beta+n+1) P^{(\alpha+1,\beta+1)}_{n-1}(x)$$

we derive the addition theorem

$$\begin{aligned} P^{(p,q)}_n(\cos 2\theta) &= \sum_{k=0}^{n} \sum_{l=0}^{k} c_{nkl} (\sin\theta_1 \sin\theta_2)^{k+l} (\cos\theta_1 \cos\theta_2)^{k-l} \\ &\times \cos^{k-l}\varphi P^{(p+k+l,q+k-l)}_{n-k}(\cos 2\theta_1) P^{(p+k+l,q+k-l)}_{n-k}(\cos 2\theta_2) \\ &\times P^{(p-q-1,q+k-l)}_l(\cos 2\varphi) C^q_{k-l}(\cos\psi), \end{aligned} \tag{4}$$

where $\cos 2\theta$ is given by formula (3′) and c_{nkl} are constants. This proof of formula (4) is due to KOORNWINDER [18]. Koornwinder gave also analytical proofs of this result [22, 24]. In [19] another group theoretical proof of formula (4) is given which uses spherical harmonics of the group $SO(n)$ with respect to the subgroup $SO(m) \times SO(n-m)$.

Both sides of formula (4) are rational functions of p and q. Therefore, this formula is correct for $p \in \mathbb{C}$ and $q \in \mathbb{C}$. Using integrations the product formula

$$P_n^{(p,q)}(\cos 2\theta_1)P_n^{(p,q)}(\cos 2\theta_2) = \frac{2\Gamma(p+1)P_n^{(p,q)}(1)}{\sqrt{\pi}\Gamma(p-q)\Gamma(q+\frac{1}{2})}$$

$$\times \int_0^1 \int_0^\pi P_n^{(p,q)}(2|\cos\theta_1\cos\theta_2 + re^{i\psi}\sin\theta_1\sin\theta_2|^2 - 1)$$

$$\times(1-r^2)^{p-q-1}r^{2q+1}\sin^{2q}\psi d\psi dr$$

is obtained [17]. By the substitution

$$e^{i\alpha}\cos\theta = \cos\theta_1\cos\theta_2 + re^{i\psi}\sin\theta_1\sin\theta_2$$

this product formula is transformed into another product formula which was obtained in a different way by GASPER [9].

Using the class 1 irreducible unitary representations of the noncompact Lie group $U(n-1,1)$ instead of representations of the group $U(n)$ the addition theorem for Jacobi functions

$$R_\mu^{(\alpha,\beta)}(x) = {}_2F_1(-\mu, \mu+\alpha+\beta+1; \alpha+1; \frac{1}{2}(1-x))$$

is obtained which is of the form

$$R_\mu^{(\alpha,\beta)}(\cosh 2t) = \sum_{m=0}^{\infty}\sum_{l=0}^{m} c_{\mu m l} r^{m-l}(\sinh t_1 \sinh t_2)^{m+l}$$

$$\times(\cosh t_1\cosh t_2)^{m-l}R_{\mu-m}^{(\alpha+m+l,\beta+m-l)}(\cosh 2t_1)R_{\mu-m}^{(\alpha+m+l,\beta+m-l)}(\cosh 2t_2)$$

$$\times P_l^{(\alpha-\beta-1,\beta+m-l)}(2r^2-1)C_{m-1}^{\beta}(\cos\psi),$$

where

$$\cosh 2t = 2|\cosh t_1\cosh t_2 + r\sinh t_1\sinh t_2\ e^{i\psi}|^2 - 1.$$

An analytical proof of this formula is given in [7]. The product formula and other interesting properties for Jacobi functions are derived in [8].

By using more general relations in the group $U(n)$ than formula (2), the addition theorem for the disc polynomials

$$R_{mn}^{\alpha}(z) = \begin{cases} P_n^{(\alpha,m-n)}(2z\bar{z}-1)z^{m-n} & if \quad m \geq n, \\ P_m^{(\alpha,n-m)}(2z\bar{z}-1)\bar{z}^{n-m} & if \quad m \leq n \end{cases}$$

is proved [22] with the help of the same matrix elements of the class 1 irreducible representations of $U(n)$. Making some manipulations and substitutions and using the limit formula

$$L_n^\alpha(x)/L_n^\alpha(0) = \lim_{\beta\to 0}\{P_n^{(\alpha,\beta)}(1-2\beta^{-1}x)/P_n^{(\alpha,\beta)}(1)\},$$

where $L_n^\alpha(x)$ is the Laguerre polynomial, KOORNWINDER [25] derived the new addition formula for Laguerre polynomials

$$\mathcal{L}_n^\alpha(x^2+y^2-2xyr\cos\psi)\exp(ixyr\sin\psi) = \sum_{k=0}^{\infty}\sum_{l=0}^{n} b_{\alpha kl}$$

$$\times(xy)^{k+l}\mathcal{L}_{n-l}^{\alpha+k+l}(x^2)\mathcal{L}_{n-l}^{\alpha+k+l}(y^2)R_{kl}^{\alpha-1}(re^{i\psi}), \tag{5}$$

where $\alpha > 0$, $n = 0, 1, 2, \ldots$, $x \geq 0$, $y \geq 0$, $0 \leq r \leq 1$, $0 \leq \psi \leq 2\pi$, and

$$\mathcal{L}_n^\alpha(x) = e^{-x/2}L_n^\alpha(x)/L_n^\alpha(0).$$

By integrating this addition formula we obtain the product formula [25]

$$\begin{aligned}\mathcal{L}_n^\alpha(x^2)\mathcal{L}_n^\alpha(y^2) &= \frac{2\alpha}{\pi}\int_0^1\int_0^\pi \mathcal{L}_n^\alpha(x^2+y^2+2xyr\cos\psi)\\ &\times\cos(xyr\sin\psi)r(1-r^2)^{\alpha-1}d\psi dr,\end{aligned}$$

where $x \geq 0$, $y \geq 0$, $\alpha > 0$.

Unfortunately, formula (5) was not directly derived by group theoretical methods. Probably, it can be obtained by an extension of results of the paper [44].

4. REPRESENTATIONS OF LIE GROUPS AND POLYNOMIALS OF DISCRETE VARIABLE

For a long time polynomials on discrete sets of points were beyond intensive investigation. These polynomials are called polynomials of a discrete variable. Most important ones are Krawtchouk, Meixner, Hahn, and Wilson polynomials (q-orthogonal polynomials are considered below). Attention to these polynomials was initiated by development of discrete mathematics, which, in turn, was stimulated by coding theory, computer science, associative schemes, design theory, and so on. The rise of interest in polynomials of a discrete variable led to application of group theoretical methods in the theory of such polynomials. Representations of Lie groups as well as those of discrete groups can be applied for studying polynomials of a discrete variable. However, the idea of applications of representations of topological groups is somewhat different.

Let T be a unitary irreducible representation of a group G in a Hilbert (finite or infinite dimensional) space L, and let $\{\mathbf{e}_j | j \in I\}$ be an orthonormal basis of L. The matrix elements $t_{mn}(g) = \langle T(g)\mathbf{e}_n, \mathbf{e}_m\rangle$ of a representation T have the property

$$\sum_{n\in I} t_{mn}(g)\overline{t_{kn}(g)} = \delta_{mk}.$$

Let us denote n by x and introduce the notation $F(x;m,g) = t_{mx}(g)$. Then for every $g \in G$ we have

$$\sum_{x\in I} F(x;m,g)\overline{F(x;k,g)} = \delta_{mk}. \tag{6}$$

Thus, at fixed $g \in G$ the set of functions $\{F(x;m,g)|m \in I\}$ constitutes an orthonormal system on I.

If l^2 is the Hilbert space of functions $f(x)$ on I with the scalar product $\langle f_1, f_2\rangle = \sum_{x\in I} f_1(x)\overline{f_2(x)}$, then any function $f \in l^2$ can be decomposed into the series

$$f(x) = \sum_{m\in I} a_m F(x;m,g), \tag{7}$$

where

$$a_m = \sum_{x\in I} f(x)\overline{F(x;m,g)}. \tag{8}$$

The corresponding Plancherel formula is valid.

Sometimes, the functions $F(x;m,g)$ are connected with known orthogonal polynomials or special functions. The formulas (7) and (8) give expansion in these polynomials or functions and formula (6) leads to an orthogonality relation for them.

This idea was firstly realized by KOORNWINDER [27] who connected matrix elements of finite dimensional irreducible representations of the group $SO(3)$ with Krawtchouk polynomials. In the same paper he also mentioned that Meixner polynomials are connected with matrix elements of discrete series representations of the group $SL(2,\mathbb{R})$ and Charlier polynomials with irreducible representations of the Heisenberg group.

Special functions of a discrete variable appear also under investigation of tensor products of irreducible representations of groups. If T_1 and T_2 are two irreducible representations of a group G, then in the space $L_1 \otimes L_2$ of the representation $T_1 \otimes T_2$ orthogonal normalized bases can be taken in two different ways. The first basis consists of tensor products of elements of bases of the spaces L_1 and L_2 and the second one is obtained if we decompose $L_1 \otimes L_2$ into irreducible subspaces and take bases in these subspaces. These two bases of $L_1 \otimes L_2$ are connected by a unitary matrix. Elements of this matrix are called Clebsch-Gordan coefficients of a group G. For the group $SU(2)$ these coefficients are expressed in terms of the function ${}_3F_2(a,b,c;d,e;1)$. KOORNWINDER [26] showed that they can also be expressed in terms of Hahn polynomials. Unitarity of the matrix, consisting of Clebsch-Gordan coefficients of $SU(2)$, easily leads to the orthogonality relation for Hahn polynomials.

This idea can be applied to representations of the group $SU(1,1)$. In particular, simple expressions for Clebsch-Gordan coefficients for tensor products of the discrete series representations of $SU(1,1)$ are obtained for different bases (elliptic, hyperbolic and parabolic ones). There are relations involving Clebsch-Gordan coefficients and matrix elements of representations. In the case of the group $SU(1,1)$ these relations lead to group theoretical interpretation of many formulas connecting different special functions and orthogonal polynomials (see Chapter 8 in [45]).

5. ROOT SYSTEMS AND GENERALIZATIONS OF HYPERGEOMETRIC FUNCTIONS

Let G be a connected real noncompact semisimple Lie group and let $\mathfrak{g}$ be its Lie algebra. We fix a Cartan decomposition $G = KAK$ of G where K is a maximal compact subgroup of G and A is a commutative subgroup. There is the commutative Lie subalgebra $\mathfrak{a}$ in $\mathfrak{g}$ such that $A = \exp \mathfrak{a}$. The dimension of A (and of $\mathfrak{a}$) is called the real rank of G. Let Σ be the root system of the pair $(\mathfrak{g}, \mathfrak{a})$ and let W be its Weyl group. The root system consists of orbits of the Weyl group W. There exist three possibilities: the root system Σ consists of one, two or three W-orbits. It is well known that multiplicities of roots in a fixed orbit are fixed. We denote multiplicities by $m_\alpha, m_\beta, m_\gamma$, where α, β, γ are roots from different orbits. If there are only two (one) orbits, then $m_\gamma = 0$ ($m_\beta = m_\gamma = 0$).

Zonal spherical functions φ of irreducible representations of the group G are, in fact, functions on A, that is $\varphi(k_1 a k_2) = \varphi(a)$, $k_1, k_2 \in K$, $a \in A$. Let $\Delta_1, \ldots, \Delta_n$ be independent Casimir differential operators on G and let $\Delta_1^0, \ldots, \Delta_n^0$ be their radial parts. The number n coincides with the rank of G. The operators Δ_i^0, $i = 1, 2, \ldots, n$, are invariant with respect to the Weyl group W. A zonal spherical function φ is solution of the system of differential equations

$$\Delta_i^0 \varphi = \mu_i \varphi, \qquad i = 1, 2, \ldots, n,$$

under appropriate numbers μ_i. Moreover, these systems uniquely determine zonal spherical functions if we demand that
(a) φ is symmetric with respect to W,
(b) φ is a regular function in the unit e of G,
(c) $\varphi(e) = 1$.

For example, if the real rank of G is equal to 1, then there exists only one operator Δ_i^0 (it coincides with the radial part of the Laplace-Beltrami operator). In this case $m_\gamma = 0$ and $\beta = 2\alpha$. Thus, we have only the multiplicities m_α and $m_{2\alpha}$. The operator Δ_1^0 difines the differential equation

$$\{\frac{d^2}{dt^2} + (m_\alpha \coth t + 2m_{2\alpha} \coth 2t)\frac{d}{dt}\}\varphi = -[\lambda(H)^2 + \rho(H)^2]\varphi, \tag{9}$$

where $a_t \equiv \exp tH \in A$, $\alpha(H) = 1$, λ is a linear form on $\mathfrak{a}$, and ρ is the half-sum of positive roots: $2\rho = (m_\alpha + 2m_{1\alpha})\alpha$. The solution of equation (9), which satisfies all necessary conditions, is

$$\varphi_\lambda(t) = c\cosh^l t\, {}_2F_1(-\frac{l}{2}, -\frac{l+m_{2\alpha}-1}{2}; \frac{m_\alpha+m_{2\alpha}+1}{2}; \tanh^2 t), \qquad (10)$$

where $l = (i\lambda - \rho)(H)$ and c is a constant. If $\mathrm{Re}(i\lambda, \alpha) > 0$, then

$$e^{(-i\lambda+\rho)(tH)}\varphi_\lambda(t) \to \mathbf{c}(\lambda) \text{ as } t \to +\infty$$

where $\mathbf{c}(\lambda)$ is the Harish-Chandra $\mathbf{c}$-function.

There are only four types of groups G with real rank 1:

for the first type	$m_\alpha \in \mathbb{Z}_+,\ m_{2\alpha} = 0,$
for the second type	$m_\alpha \in \mathbb{Z}_+,\ m_{2\alpha} = 1,$
for the third type	$m_\alpha \in \mathbb{Z}_+,\ m_{2\alpha} = 3,$
for the fourth type	$m_\alpha = 8,\ m_{2\alpha} = 7.$

Since $m_\alpha \in \mathbb{Z}_+$ and $m_{2\alpha}$ takes one of the values 0,1,3,7, then zonal spherical functions (10) can not give a general hypergeometric function ${}_2F_1$. A similar situation takes place in the general case since we have certain restrictions for values of the multiplicities m_α, m_β, m_γ.

Let us note that if one of the numbers $l/2$, $(l + m_{2\alpha} - 1)/2$ in (10) is a positive integer, then this hypergeometric series is terminating and φ_λ is expressed in terms of the Jacobi polynomial. This statement is valid for rank $n > 1$. Namely, if parameters determining zonal spherical functions φ satisfy some integrality conditions, then φ can be expressed as a polynomial of n variables (generalized Jacobi polynomials).

In order to obtain in (10) a general hypergeometric function ${}_2F_1$ and a general Jacobi polynomial, one allows for m_α and $m_{2\alpha}$ to take any complex values (except for some singular points). In this way we turn to special functions connected with root systems. Namely, in the case of the root system corresponding to real rank 1 we consider W-invariant differential equation (9), where m_α and $m_{2\alpha}$ are fixed complex numbers (except for the singular points). A solution of this equation, regular at the point $t = 0$, gives a general hypergeometric function ${}_2F_1$ (a general Jacobi polynomial if the integrality conditions are fulfilled).

These considerations were given for real rank 2 by KOORNWINDER [20, 23]. He obtained Jacobi polynomials of two variables corresponding to the root systems A_2 and BC_2. A further generalization was given by Vratare [46]. He defined the generalized Jacobi polynomials $p_m^{(a,b,c)}(x)$, where a, b, c are determined by m_α, m_β, m_γ and $x = (x_1, \ldots, x_n)$, $m = (m_1, \ldots, m_n)$, $m_i \in \mathbb{Z}_+$. Generalized hypergeometric functions related to root systems with complex multiplicities were given by HECKMAN and OPDAM [13]. The case of root system BC_n is considered by DEBIARD [4].

Of course, it is necessary to study Heckman-Opdam hypergeometric functions and to establish which properties of zonal spherical functions remain valid for them. Let us make the following remark related to this problem. We know that the spherical transform

$$F(\lambda) = \int_G f(g)\varphi_{i\lambda-\rho}(g)dg \qquad (11)$$

is defined with the help of zonal spherical functions $\varphi_{i\lambda-\rho}$ corresponding to unitary irreducible representations of G. The inverse transform is of the form

$$f(g) = |W| \int_{\Omega} F(\lambda)\overline{\varphi_{i\lambda-\rho}(g)}|\mathbf{c}(\lambda)|^{-2}d\lambda, \tag{12}$$

where $\mathbf{c}(\lambda)$ is the Harish-Chandra $\mathbf{c}$-function and Ω is the set of values taken by parameters, giving non-equivalent irreducible representations of the spherical principal unitary series. The corresponding Plancherel formula is valid. It was shown by Flensted-Jensen and Koornwinder (see, for example, [21]) that in the case of the root system of real rank 1 formulas (11) and (12) remain valid if we replace multiplicities m_α and $m_{2\alpha}$ by complex multiplicities. Here we obtain the Harish-Chandra $\mathbf{c}$-function $\mathbf{c}(\lambda)$ and the corresponding Plancherel measure related to the root system with complex m_α and $m_{2\alpha}$. A similar analytical continuation of spherical transforms (11) and (12) for real rank $n > 1$ was not obtained. In the real rank 1 case this analytical continuation, in fact, coincides with a well-known Jacobi transform.

6. Special functions of a matrix argument

In 1955 Herz [14] introduced the hypergeometric function ${}_pF_q$ of a matrix argument using Laplace and inverse Laplace transforms of functions of a matrix argument. This idea was originated by Bochner who considered Bessel functions of a matrix argument. Constantine [3] found a series expansion for Herz's ${}_pF_q$ in terms of zonal polynomials. These zonal polynomials are the spherical functions of certain irreducible polynomial representations of the group $GL(n,\mathbb{R})$. Recently Gross and Richards [10] treated hypergeometric functions ${}_pF_q$ with zonal polynomials for the groups $GL(n,\mathbb{F})$, $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$. There are further generalizations of these hypergeometric functions (for example, ones obtained by replacement of zonal polynomials by Jack polynomials).

Let $\mathbb{F}$ be one of the fields $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and let $\mathfrak{h}(n,\mathbb{F})$ be the space of all Hermitian $n\times n$ matrices Λ over $\mathbb{F}$. The formula $\Lambda \to g^*\Lambda g$, $g \in GL(n,\mathbb{F})$, defines an action of $GL(n,\mathbb{F})$ in $\mathfrak{h}(n,\mathbb{F})$. Let $\mathcal{P}(\mathfrak{h})$ be the algebra of all polynomial functions on $\mathfrak{h}(n,\mathbb{F})$. Then the action of $GL(n,\mathbb{F})$ in $\mathcal{P}(\mathfrak{h})$ is defined which gives reducible representation T of this group. This representation in $\mathcal{P}(\mathfrak{h})$ is multiplicity free. We have the decomposition

$$\mathcal{P}(\mathfrak{h}) = \sum_{\mathbf{m}} \oplus \mathcal{P}_{\mathbf{m}}(\mathfrak{h})$$

of $\mathcal{P}(\mathfrak{h})$ into irreducible subspaces, where $\mathbf{m} = (m_1, \ldots, m_n)$, $m_1 \geq m_2 \geq \ldots \geq m_n \geq 0$, $m_i \in \mathbb{Z}$, determine highest weights of the corresponding irreducible representations. Polynomials of $\mathcal{P}_{\mathbf{m}}(\mathfrak{h})$ are homogeneous of degree $m \equiv |\mathbf{m}| = m_1 + \ldots + m_n$. In every subspace $\mathcal{P}_{\mathbf{m}}(\mathfrak{h})$ there is a unique (up to a constant) K-invariant polynomial $Z_{\mathbf{m}}(\Lambda)$, where K is the maximal compact subgroup in $GL(n,\mathbb{F})$. We refer to $Z_{\mathbf{m}}$ as the zonal polynomial on $\mathfrak{h}(n,\mathbb{F})$ of weight $\mathbf{m}$. It is clear that these polynomials are related to zonal spherical functions of the corresponding representations of $GL(n, \mathbb{F})$ with respect to the subgroup K. We normalize the polynomials $Z_{\mathbf{m}}(\Lambda)$ such that

$$(\mathrm{Tr}\Lambda)^r = \sum_{|\mathbf{m}|=r} Z_{\mathbf{m}}(\Lambda).$$

Then the hypergeometric function ${}_pF_q$ of a matrix argument is defined to be the real-analytical function on $\mathfrak{h}(n, \mathbb{F})$ given by the series

$$\begin{aligned} &{}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; \Lambda) \\ &= \sum_{r=0}^{\infty} \sum_{|\mathbf{m}|=r} \frac{[a_1]_{\mathbf{m}} \ldots [a_p]_{\mathbf{m}}}{[b_1]_{\mathbf{m}} \ldots [b_q]_{\mathbf{m}}} \frac{Z_{\mathbf{m}}(\Lambda)}{r!}, \end{aligned} \tag{13}$$

where $[a]_{\mathbf{m}}$ is the generalization of the classical Pochhammer symbol:

$$[a]_{\mathbf{m}} = \prod_{j=1}^{n} (a - \frac{1}{2}(j-1)\nu)_{m_j}.$$

Here ν denotes the real dimension of $\mathbb{F}$ and $(a)_j = a(a+1)\ldots(a+j-1)$. If $p \le q$ then the series (13) converges absolutely for all $\Lambda \in \mathfrak{h}(n, \mathbb{F})$. If $p = q+1$ then it converges absolutely for $\| \Lambda \| < 1$, where $\| \Lambda \| = \max\{\lambda_j | j = 1, 2, \ldots, n\}$ and $\lambda_1, \ldots, \lambda_n$ are eigenvalues of Λ. If $p > q$ then the series (13) diverges unless it terminates. We have

$${}_0F_0(\Lambda) = \exp(\mathrm{Tr}\Lambda), \quad {}_1F_0(a; \Lambda) = \Delta(I - \Lambda)^{-a},$$

where I is the unit $n \times n$ matrix and $\Delta(\Lambda)$ denotes determinant of a matrix Λ. For the hypergeometric function ${}_2F_1$ of a matrix argument the formula

$$\begin{aligned} &{}_2F_1(a, b; c; \Lambda) \\ &= \frac{\Gamma_n(c)}{\Gamma_n(b)\Gamma_n(c-b)} \int_0^I \Delta(M)^{b-\theta} \Delta(I-M)^{c-b-\theta} \Delta(I - M\Lambda)^{-a} dM \end{aligned}$$

is valid, where $\theta = \frac{1}{2}(n-1)\nu + 1$ and the generalized Γ-function $\Gamma_n(a)$ is defined as

$$\Gamma_n(a) = \int_0^{\infty} \exp(-\mathrm{Tr}\Lambda) \Delta(\Lambda)^{a-\theta} d\Lambda$$

(the integration is over all positive definite matrices from $\mathfrak{h}(n, \mathbb{F})$).

KORÁNYI [34] and MACDONALD [35] generalized hypergeometric functions (13) by replacement of $Z_m(\Lambda)$ by the appropriately normalized Jack polynomials. Let us note that zonal polynomials are obtained from Jack polynomials by fixing appropriately one of its parameters. BEERENDS and OPDAM [2] showed that Jack polynomials are related to generalized Jacobi polynomials associated with the root system A_n.

KOORNWINDER [23] studied connections of his generalized two-variables Jacobi polynomials with the hypergeometric function ${}_2F_1$ of a matrix argument. BEERENDS and OPDAM [2] generalized these studies. They showed that the generalized hypergeometric function ${}_2F_1$ of a matrix argument can be obtained as a special case of the hypergeometric function associated with the root system BC_n. Extensive study of the BC_2 Jacobi polynomials is given by KOORNWINDER and SPRINKHUIZEN-KUYPER [32]. They derived relations of these polynomials with zonal polynomials, Appell's F_4 hypergeometric function, and a particular function of order 3. In [33] the same authors show that the hypergeometric function of 2×2-matrix argument is expressible as a solution of the partial differential equation for Appell's F_4 function. As a result, the first function can be written as a sum of two F_4 functions.

7. THE QUANTUM GROUP $SU_q(2)$ AND q-ORTHOGONAL POLYNOMIALS

Basic hypergeometric functions and their partial and limiting cases constitute large class of special functions depending on the additional parameter q. The corresponding orthogonal polynomials are called q-orthogonal polynomials. q-Krawtchouk, q-Hahn, dual q-Hahn, q-Askey-Wilson, little q-Jacobi, continuous q-Jacobi, big q-Jacobi, q-ultraspherical, q-Laguerre, q-Hermite polynomials belong to widespread q-orthogonal polynomials. These polynomials can also be investigated by group theoretical methods. It was shown [38-40] that q-orthogonal polynomials with $q = p^s$, where p is a primitive number and s is a positive integer, are connected with irreducible representations of Chevalley groups. Namely, zonal spherical functions and intertwining functions of these representations are expressed in terms of these polynomials. Using these groups, addition and product formulas, orthogonality relations and other properties of these q-orthogonal polynomials can be derived.

It was shown recently that q-orthogonal polynomials are related to representations of quantum groups. Representations of quantum groups describe q-orthogonal polynomials for any complex q. Quantum groups appeared in the quantum method of the inverse scattering problem. They are more complicated mathematical objects than discrete or topological groups. Quantum groups are not manifolds. A quantum group is defined with the help of algebra of functions on it which is a Hopf algebra [12, 29].

Basic hypergeometric functions are defined in terms of the expression

$$(a;q)_n = \prod_{j=0}^{n-1} (1 - aq^j), \quad a \in \mathbb{C}, \ n \in \mathbb{Z}_+.$$

For $n = 0$ we set $(a;q)_0 = 1$. We have

$$\lim_{q \to 1} \frac{(q^a;q)_n}{(q^b;q)_n} = \frac{(a)_n}{(b)_n} \text{ where } (a)_n = \Gamma(a+n)/\Gamma(a).$$

Basic hypergeometric functions (or q-hypergeometric series) are given by the formula

$$
{}_{n+1}\varphi_n(a_1,\ldots,a_{n+1};b_1,\ldots,b_n;q,z)
$$

$$
= \sum_{r=0}^{\infty} \frac{(a_1;q)_r \ldots (a_{n+1};q)_r}{(b_1;q)_r \ldots (b_n;q)_r} \frac{z^r}{(q;q)_r}. \tag{14}
$$

The radius of convergence for generic values of the parameters is equal to 1. If $a_1 = q^{-n}$, $n \in \mathbb{Z}_+$, and $b_1, \ldots, b_n \neq 1, q^{-1}, \ldots, q^{-n}$, then series (14) is terminating and well defined. In these cases we obtain polynomials. The polynomials

$$
p_n(x;a,b|q) = {}_2\varphi_1(q^{-n}, abq^{n+1}; aq; q, qx)
$$

are called the little q-Jacobi polynomials. The formula

$$
Q_n(x;a,b,N|q)
$$

$$
= {}_3\varphi_2(q^{-n}, abq^{n+1}, x; aq, q^{-N}; q, q),
$$

where $N \in \mathbb{Z}_+$ and $n \in \{0,1,\ldots,N\}$, defines the q-Hahn polynomials. The dual q-Hahn polynomials also can be defined. The polynomials

$$
K_n(x;b,N|q) = {}_2\varphi_1(q^{-n}, x; q^{-N}; q, bq^{n+1}),
$$

where $N \in \mathbb{Z}_+$ and $n \in \{0,1,\ldots,N\}$, are called the q-Krawtchouk polynomials. There exist three other types of q-Krawtchouk polynomials [40]. The formula

$$
p_n(\cos\theta; a,b,c,d|q) = a^{-n}(ab;q)_n(ac;q)_n(ad;q)_n
$$

$$
\times {}_4\varphi_3(q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta}; ab, ac, ad; q, q)
$$

gives the q-Askey-Wilson polynomials. They are symmetric in the parameters a,b,c,d [1].

Let us define the algebra of functions $A(SU_q(2))$ on the quantum group $SU_q(2)$. We consider the associative algebra A' generated by the elements x,u,v,y which obey the relations

$$
ux = \sqrt{q}\,xu, \quad vx = \sqrt{q}\,xv, \quad yu = \sqrt{q}\,uy, \quad yv = \sqrt{q}\,vy,
$$

$$
vu = uv, \quad xy - q^{-1/2}uv = yx - \sqrt{q}\,uv = 1,
$$

where q is a fixed complex number which is not a root of unity. The structure of a Hopf algebra is introduced into A', that is, the operation of comultiplication $\Delta : A' \to A' \otimes A'$, the counit $\varepsilon : A' \to \mathbb{C}$, and the antipode $S : A' \to A'$ are defined. The comultiplication Δ and the counit ε are algebra homomorphisms and are uniquely determined by the relations

$$
\Delta \begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} x\otimes x + u\otimes v & x\otimes u + u\otimes y \\ v\otimes x + y\otimes v & v\otimes u + y\otimes y \end{pmatrix},
$$

$$
\varepsilon \begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

where elementwise equalities are understood. The antipode S is an algebra anti-homomorphism and is determined by the equalities

$$S\begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} y & -\sqrt{q}\,u \\ -q^{-1/2}v & x \end{pmatrix}.$$

The associative algebra A' with the structure of a Hopf algebra introduced is called the algebra of functions on the quantum group $SL_q(2,\mathbb{C})$. It is denoted by $A \equiv A(SL_q(2,\mathbb{C}))$.

If the parameter q is real, then we can introduce $\star$-operation into $A(SL_q(2,\mathbb{C}))$. This operation is a conjugate linear ring antihomomorphism $a \to a^*$, $a \in A$, and, therefore, is uniquely determined by the formula

$$\begin{pmatrix} x & u \\ v & y \end{pmatrix}^* \equiv \begin{pmatrix} x^* & v^* \\ u^* & y^* \end{pmatrix} = \begin{pmatrix} y & -\sqrt{q}\,u \\ -q^{-1/2}v & x \end{pmatrix}.$$

The Hopf algebra $A(SL_q(2,\mathbb{C}))$ with this $\star$-operation is a $\star$-Hopf algebra. It is called the algebra of functions on the quantum group $SU_q(2)$ and is denoted by $A(SU_q(2))$. The quantum group $SU_q(2)$ is said to be a compact real form of the quantum group $SL_q(2,\mathbb{C})$.

Let V be a linear space. A linear mapping $T: V \to V \otimes A$ is called a right corepresentation of the Hopf algebra $A \equiv A(SL_q(2,\mathbb{C}))$ in V if

$$(\mathrm{id} \otimes \varepsilon) \circ T = \mathrm{id}, \quad (\mathrm{id} \otimes \Delta) \circ T = (T \otimes \mathrm{id}) \circ T.$$

The space V is said to be a right A-comodule. A linear mapping $T: V \to A \otimes V$ is called a left corepresentation of A in V if

$$(\varepsilon \otimes \mathrm{id}) \circ T = \mathrm{id}, \quad (\Delta \otimes \mathrm{id}) \circ T = (\mathrm{id} \otimes T) \circ T.$$

The space V in this case is a left A-comodule.

Let $T: V \to V \otimes A$ be a right corepresentation of A. If W is a subspace of V such that $T(W) \subset W \otimes A$, then W is said to be a right A-subcomodule and the mapping $T: W \to W \otimes A$ is a right subcorepresentation of the corepresentation $T: V \to V \otimes A$. One can easily formulate definitions of a direct (orthogonal) sum of right (left) corepresentations, of irreducibility and of complete irreducibility of corepresentations.

Let $T: V \to V \otimes A$ and $Q: W \to W \otimes A$ be right corepresentations of $A \equiv A(SL_q(2,\mathbb{C}))$. If there exists a linear invertible mapping $F: V \to W$ such that

$$(Q \circ F)v = ((F \otimes \mathrm{id}) \circ T)v \quad \text{for all } v \in V,$$

then T and Q are equivalent. In the same way the notion of equivalence of left corepresentations is given.

Let $e_1, \ldots, e_n$ be a basis in a right A-comodule V, where a corepresentation T is realized. Then

$$Te_i = \sum_j e_j \otimes t_{ji}, \quad \text{where } t_{ji} \in A.$$

The elements t_{ji} of the algebra $A(SL_q(2,\mathbb{C}))$ are called matrix elements of the corepresentation T. It is easy to prove that

$$\Delta(t_{ji}) = \sum_{k=1}^{n} t_{jk} \otimes t_{ki}.$$

In the same way matrix elements of a left corepresentation are defined.

Let l be a positive integer or half-integer. We construct the linear spaces V_l^L and V_l^R spanned by the elements

$$e_i = \left[\begin{array}{c} 2l \\ l+i \end{array} \right]_q^{1/2} x^{l-i} v^{l+i}, \quad i = -l, -l+1, \ldots, l, \tag{15}$$

and by the elements

$$f_i = \left[\begin{array}{c} 2l \\ l+i \end{array} \right]_q^{1/2} x^{l-i} u^{l+i}, \quad i = -l, -l+1, \ldots, l, \tag{16}$$

respectively, where $\left[\begin{array}{c} n \\ m \end{array} \right]_q = (q;q)_n [(q;q)_m (q;q)_{n-m}]^{-1}$. It is proved [36] that

$$\Delta : V_l^R \to V_l^R \otimes A, \quad \Delta : V_l^L \to A \otimes V_l^L.$$

This means that the comultiplication Δ determines a left corepresentation of A in V_l^L and a right corepresentation of A in V_l^L. They are denoted by T_l^L and T_l^R respectively. Elements (15) form a basis in V_l^L and elements (16) form a basis in V_l^R. We have

$$\Delta e_i = \sum_{j=-l}^{l} t_{ij}^l \otimes e_j, \quad \Delta f_i = \sum_{j=-l}^{l} f_j \otimes d_{ji}^l.$$

It is proved [36] that $t_{ij}^l = d_{ij}^l$ for all l, i, j. The matrix elements are expressed in terms of the basic hypergeometric function ${}_2\varphi_1$ which is a little q-Jacobi polynomial [28, 36, 42]. We have

$$t_{ij}^l = a_{i,-j} q^{(l+i)(j-i)/2} x^{-i-j} v^{i-j} p_{l+j}(\zeta; q^{i-j}, q^{-i-j}|q) \text{ if } i+j \le 0,\ i \ge j,$$

$$t_{ij}^l = a_{-i,j} q^{(l+i)(i-j)/2} x^{-i-j} u^{j-i} p_{l+i}(\zeta; q^{j-i}, q^{-i-j}|q) \text{ if } i+j \le 0,\ j \ge i,$$

$$t_{ij}^l = a_{-i,j} q^{(j-i)(j-l)/2} p_{l-j}(\zeta; q^{j-i}, q^{i+j}|q) u^{j-i} y^{i+j} \text{ if } i+j \ge 0,\ j \ge i,$$

$$t_{ij}^l = a_{i,-j} q^{(i-j)(i-l)/2} p_{l-i}(\zeta; q^{i-j}, q^{i+j}|q) v^{i-j} y^{i+j} \text{ if } i+j \ge 0,\ i \ge j,$$

where $\zeta = -q^{-1/2} uv$ and

$$a_{i,j} = \left[\begin{array}{c} l+i \\ i+j \end{array} \right]_q^{1/2} \left[\begin{array}{c} l+j \\ i+j \end{array} \right]_q^{1/2}.$$

Using these expressions for t^l_{ij}, KOORNWINDER [31] proved the addition formula for little q-Legendre polynomials $p_l(q^z;1,1|q)$. Let us note that we do not have one-parameter subgroups in quantum groups as in the case of classical groups. Therefore, to derive addition theorems we can not use relations between products of one-parameter subgroups (such relations are used to prove addition theorems for classical orthogonal polynomials and special functions). To derive the addition theorem for little q-Legendre polynomials, Koornwinder used representations of the Hopf algebra $A(SU_q(2))$. A representation of a Hopf algebra B is a representation of an associative algebra B.

In paper [28] Koornwinder expressed the matrix elements t^l_{ij} in terms of the q-Krawtchouk polynomials $K_n(x;b,N|q)$. Unitarity of the corepresentations T_l of the Hopf algebra $A(SU_q(2))$ means that

$$\sum_{n=-l}^{l} (t^l_{nk})^* t^l_{nm} = \delta_{km} I,$$

where I is the unit of $A(SU_q(2))$. This formula easily leads to the orthogonality relation for q-Krawtchouk polynomials. The orthogonality relation for q-Jacobi polynomials follows from the orthogonality relations for the matrix elements t^l_{mn} (see, for example, [36]).

Elements t^l_{00} and t^l_{m0} are zonal and associated spherical functions of the quantum group $SU_q(2)$ with respect to the quantum subgroup H_q which is the quantum analogue of the one-parameter diagonal subgroup H of the Lie group $SU(2)$. Other one-parameter subgroups of $SU(2)$ are conjugate to the subgroup H. For this reason other subgroups do not give new results for orthogonal polynomials. This is not the case for the quantum group $SU_q(2)$. For example, the quantum group $SU_q(2)$ does not have a quantum analogue of the subgroup $SO(2)$ of the group $SU(2)$. In order to consider q-analogues of spherical functions of the group $SU(2)$ with respect to subgroups different from H, KOORNWINDER [30] developed an infinitesimal approach to spherical functions of the quantum group $SU_q(2)$. Namely, for $SU_q(2)$ he gave the notion of spherical elements by considering left and right invariance in the infinitesimal sense with respect to twisted primitive elements of the quantized universal enveloping algebra $U_q(\mathfrak{sl}\,(2))$. The resulting spherical elements (which belong to the Hopf algebra $A(SU_q(2))$ and correspond to the irreducible corepresentations T_l) turn out to be expressible as a two-parameter family of q-Askey-Wilson polynomials. By using this approach KOORNWINDER [30] also obtains dual q-Krawtchouk polynomials. A further development of this approach is given by H.T. Koelink and by M. Noumi and K. Mimachi. They obtained more general results for q-Askey-Wilson polynomials (including the addition formulas).

Clebsch–Gordan coefficients of tensor products $T_{l_1} \otimes T_{l_2}$ of matrix corepresentations $T_l \equiv (t^l_{mn})$ of the Hopf algebra $A(SL_q(2,\mathbb{C}))$ are defined in the same way as in the case of the Lie group $SU(2)$. They are entries of the numerical matrix which transforms the tensor product basis into the basis cosisting of bases of irreducible subcomodules. KOELINK and KOORNWINDER [15] connected these

Clebsch-Gordan coefficients with q-Hahn polynomials $Q_n(x; a, b, N|q)$ and with dual q-Hahn polynomials. The orthogonality relations for Clebsch-Gordan coefficients lead to the orthogonality relations for these polynomials. For generating functions and some other properties of Clebsch-Gordan coefficients of $SU_q(2)$ we refer also to [11].

8. Other quantum groups and q-special functions

q-Orthogonal polynomials can be studied by means of other quantum groups. Noumi, Yamada, and Mimachi [37] proved the q-analogue of the Peter-Weyl theorem for matrix elements of irreducible representations of the quantum group $U_q(n)$ and derived expressions for zonal spherical functions on the quantum space $U_q(n)/U_q(n-1)$.

Spherical functions on the quantum spheres $S_q^2(c, d)$ and $S_q^3(c, d)$ (which are introduced by means of algebras of functions) are expressed in terms of big q-Jacobi polynomials and q-Hahn polynomials. M. Noumi and K. Mimachi connected the quantum sphere $S_q^2(1, 1)$ with continuous q-ultraspherical polynomials $C_N^\lambda(x; q)$.

The quantum group of plane motions is used to investigate q-Bessel functions. Representations of the noncompact quantum group $SU_q(1, 1)$ lead to the basic hypergeometric function ${}_2\varphi_1$. In particular, representations of the discrete series give little q-Jacobi polynomials and q-Meixner polynomials.

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